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Classical Diffusion:  
Theory and Simulation Codes

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## Preface

This report is the written version of a talk presented by William Grossmann at the Varenna Workshop on High Beta Plasmas at Varenna, Italy, September 1977. It presents some of the highlights of the extensive theoretical and numerical program on plasma equilibrium, transport, and adiabatic processes, including an outline of some of the simulation codes which have been developed to implement this body of work.

### Abstract

A survey is given of the development of classical diffusion theory which arose from the observation of Grad and Hogan that the Pfirsch-Schlüter and Neoclassical theories are very special and frequently inapplicable because they require that plasma mass flow be treated as transport rather than as a state variable of the plasma. The subsequent theory, efficient numerical algorithms, and results of various operating codes are described.

The theory of classical diffusion in toroidal (or periodic) systems originated with Kruskal and Kulsrud in 1958 [1]. They took a form of Ohm's law equivalent to

$$(1) \quad E + u \times B = \eta J$$

and assumed that the system was fully stationary in time [ $\rho du/dt$  dropped from momentum conservation,  $\partial B/\partial t$  from flux conservation, and  $\partial \rho/\partial t$  from mass conservation - except that an artificial mass source was inserted in order to avoid the consequent restriction to only trivial solutions]. Pfirsch and Schlüter [2] made this theory more useful by evaluating explicit formulas for the diffusion rate in the special case of large aspect and approximately circular flux surfaces. Some time later these formulas were generalized by Maschke [3] to include  $\oint E \cdot dx \neq 0$  (consistent with  $\text{curl } E = 0$  and  $\partial B/\partial t = 0$  in the plasma itself) in order to represent the Tokamak configuration.

There are two outstanding mysteries which arise in this "classical" theory; these were pointed out (and resolved) in 1970 by Grad and Hogan [4] and Grad [5]. A velocity field  $u(x,y,z)$  is set up by the resistivity (in what would be a static equilibrium, ideally) but only the net flow through a flux surface

$$(2) \quad U = \oint u \cdot ds = \langle u \cdot \nabla p \rangle / p' (V) = \langle u \cdot \nabla V \rangle \quad , \quad V = \text{Volume}$$

is of major interest. Surprise number one is that the net

flow can be evaluated directly without having to first compute the entire flow field  $u(x,y,z)$ . Multiplying (1) by  $J$  and averaging over a flux surface,

$$(3) \quad \langle E \cdot J \rangle - \langle u \cdot J \times B \rangle = \langle \eta J^2 \rangle .$$

By assumption

$$(4) \quad \nabla p = J \times B$$

$$(5) \quad \text{curl } E = 0$$

from which the first term in (3) vanishes and we obtain

$$(6) \quad U = \langle \eta J^2 \rangle / p'(V)$$

which is, in principle, computable in terms of a given static equilibrium. If  $J \cdot B = 0$ , then  $J^2 = |\nabla p|^2 / B^2$ , and the classical  $1/B^2$  formula results

$$(7) \quad U = - \left\langle \eta \frac{\nabla p \cdot \nabla v}{B^2} \right\rangle .$$

The extra contribution from  $J_{||}$  with approximately circular cross-section gives the familiar Pfirsch-Schlüter factor.

The second, more disturbing consequence of the Kruskal, Kulsrud, Pfirsch, Schlüter ( $K^2PS$ ) theory is the fact that this model severely restricts the relevant equilibria. In a general toroidal configuration (ignoring the very real questions of a nonexistence of equilibria through topological instability, Grad 1967 [6]), one should be able to specify two essentially

arbitrary profiles (cf. Grad and Rubin [7] or Kruskal and Kulsrud [1]). The  $\kappa^2$ PS diffusion theory and formulas apply only to a special subclass of equilibria for which

$$(8) \quad \langle \eta \mathbf{J} \cdot \mathbf{B} \rangle = 0$$

on each surface (an immediate consequence of (1), after multiplication by  $\mathbf{B}$  and averaging). For the Tokamak,

$$\oint \mathbf{E} \cdot d\mathbf{x} = c \neq 0, \text{ the restriction}$$

$$(9) \quad c\psi'(V) = \langle \eta \mathbf{J} \cdot \mathbf{B} \rangle$$

also allows only one profile to be freely specified.

The two questions,

- i) Why is the profile restricted, and what is the diffusion rate in general, without this restriction?
- ii) What is the source of the miracle, that the net flow can be calculated without first solving a boundary value problem for  $u(x,y,z)$ , and does it persist in the general case?

are resolved by dropping  $\text{curl } \mathbf{E} = 0$  and re-introducing  $\partial \mathbf{B} / \partial t$ . The extension to  $\text{curl } \mathbf{E} \neq 0$  turns out to allow general equilibrium profiles, gives entirely different "paleo-classical" diffusion results, and requires solution of a global boundary value problem before obtaining the desired surface averages. It also introduces coupling between plasma diffusion and another classical phenomenon, the skin effect.



Question (ii) above is answered in greater generality in [5]. Briefly, in an irreversible thermodynamic formulation

$$(10) \quad v_i = A_{ij} u_j$$

in which the fluxes  $v_i(x)$ , forces  $u_i(x)$  and transport coefficients  $A_{ij}(x)$  depend on a parameter  $x$ . The net flux  $\int v_i dx$  can be a linear function of the forces with a universal matrix which is independent of the values of  $u_i$  only when either the  $u_i$  or  $v_i$  are independent of  $x$  (in which case the simplification is evident). In the plasma diffusion example, the facts that the "force"  $p$  [more accurately,  $p'(V)$ ] is constant on the flux surface, and the force (actually flux!)  $\oint E \cdot dx$  is path independent, are what allow surface averaged transport coefficients. Similarly, with heat flow the assumption that  $T$  is constant on a flux surface must be made. Without these assumptions (e.g. with anisotropic stress tensor, or  $T \neq T(\psi)$ , or  $\text{curl } E \neq 0$ ) there are no global transport coefficients relating surface averaged fluxes; the entire time-dependent, two or three dimensional flow problem must be solved, after which averages can be computed; (however, by various ingenious stratagems and several years of development, the total amount of effort required to solve the more general transport problems is not much more than for the restricted  $K^2PS$  special case).

The mathematical and numerical problem after reinsertion of  $\partial B / \partial t$  is very nonstandard. The reason is found by comparing

this Grad-Hogan (GH) model with the full MHD system including  $\rho du/dt$  as well as  $\partial B/\partial t$ . Instead of marching  $u$  in time, the GH system states that  $u$  must be determined by the constraint that  $B$  and  $p$  continue to satisfy  $J \times B = \nabla p$  as they vary in time. The canonical procedure for obtaining the equation governing  $u$  is to differentiate the constraint (i.e. the pressure balance) with respect to  $t$ , replacing  $\partial p/\partial t$ ,  $\partial B/\partial t$ ,  $\partial J/\partial t$  with expressions involving  $u$ . The resulting monstrous equation for  $u$  is nonstandard, and is not a differential equation. In 2-D, axial symmetry, and helical symmetry it is a linear GDE (originally QDE), for which substantial theory and effective numerical algorithms have been developed since 1970 (theory [8-10], numerical [11-16]). There is a computational tradeoff between taking a conventional system of differential equations, including  $\rho du/dt$ , with its very wide span of physical time scales, or taking the less conventional GDE on a much more manageable transport time scale. Because of the substantial analytical and numerical developments, Refs. [8-18], the correct decision would seem to be in favor of GH rather than full MHD with  $\rho du/dt$ , except for problems of plasma initiation (usually on microsecond time scales).

The key to developing accurate, fast, efficient plasma transport codes turned out to depend on techniques developed for the adiabatic problem. The two problems are mathematically similar in that each follows a family of static equilibria. The very successful adiabatic algorithm depends on alternating

between 1-D profile determination (from an ODE with tentative inductance coefficients) and 2-D contour determination [from an elliptic PDE with tentative current profile  $J(V)$ ]. The contour shapes determine the inductance profiles, and the average pressure balance (containing inductance and given adiabatic profiles) determines the correct profile [9,11]. Among the important numerical features are the use of  $\Delta\psi = J(V)$  instead of  $\Delta\psi = J(\psi)$  to improve numerical stability, and application of analytical knowledge of the singular behavior near the plasma center ( $V = 0$ ), plasma edge, and separatrix (different for adiabatic and resistive, symmetric and asymmetric cases, etc.) which allows much more efficient computation.

A prototype adiabatic GDE is

$$(11) \quad \Delta\psi = -\psi''.$$

On the left is an elliptic operator on  $\psi(x,y)$ . Writing  $V(\psi)$  for the volume (area in 2-D) within a  $\psi$ -contour, and  $\psi(V)$  for the inverse function, the right side is an ordinary differential operator on  $\psi(V)$ . Conventional methods of solution which work for the elliptic equation,  $\Delta\psi = f(\psi)$ , or even  $\Delta\psi = f(V)$  (e.g. iteration on the right side) do not work for the GDE. The iteration between 1-D and 2-D, i.e., between (11) and its average

$$(12) \quad (K\psi')' = -\psi'', \quad K(V) = \langle |\nabla V|^2 \rangle$$

is the only known practical numerical method.

There are essentially three procedures for solving non-inertial ( $\rho \frac{du}{dt} = 0$ ) transport problems. The first (1970 [8]) makes use of the Green's function for the elliptic operator which is the linearization (small variation in time  $\Delta t$ ) of the equilibrium. This is very impractical numerically (since the Green's function has to be recomputed each time step), but is very appropriate for analytical work and has led to an existence proof for the GDE involved in this problem [10] (also for the adiabatic problem).

A second procedure is to solve the linear GDE for the velocity  $u \cdot \nabla \psi$  at each time step (solved by iterating between profile and geometry), inserting the velocity to then march  $\psi$ ,  $p$ , etc. This is numerically impractical because the velocity profile exhibits boundary layers and rapid variation and would require small time steps. It is occasionally useful analytically, and was the source of the original demonstration by Grad and Hogan that in some very special cases (principally low  $\beta$ , large aspect, small poloidal field) there is a rapid (skin effect) time scale which is nonclassical during which the two independent equilibrium profiles relax until they satisfy the  $K^2$ PS profile restriction, and during which the diffusion velocity relaxes to the "classical" value. (A somewhat similar theory was repeated later [17], and has been carried out with more general transport [18]).

The third procedure for GH diffusion, the one which has proved to be most effective - so much so that a very nonstandard

2-D diffusion problem is reduced to one with computation time only slightly more than for a standard 1-D diffusion problem - is to introduce independent and dependent variables so as to eliminate the convection velocity; (it can be computed afterwards, if desired). The time steps can be taken very large because (in the proper variables) the geometric coefficients vary very slowly, even through a change in topology. In a moderately optimized resistive calculation in which a Belt Pinch is squeezed, isolates, and develops into a fully-formed Doublet, the computation time on a CDC 6600 was one minute.

The crucial point is to take the proper independent and dependent variables in the properly averaged transport and pressure balance equations. For example,  $\psi$ ,  $\chi$ ,  $M = \int \rho dV$  are possible independent variables;  $V$  is not (except at low  $\beta$ , large aspect, small poloidal field). As dependent variables,  $\psi$  and  $\chi$  are possible;  $\psi'(V)$  is not, but  $\psi'/\rho$  is satisfactory;  $p$  is not a good variable but  $p/\rho^Y$  or  $p/(\psi')^Y$  are; with reversed field,  $\psi$  or  $\chi$  is a poor independent variable. More precisely, normalized independent variables, e.g.  $V/V_{sep}$  or  $\psi/\psi_{plasma}$  are essential because the inductance coefficients (and consequently the current) have moving singularities, following  $V_{sep}$ , and a current skin layer may follow  $V_{plasma}$ .

Before writing down the mathematical formulation, there are several qualitative features to mention. The basic concept is that of a slowly varying family of equilibria (this

is common to adiabatic and dissipative models and is the cause of the mathematical and numerical similarity). In Ohm's law, (1)  $J$  and  $B$  can be taken as given (they define the instantaneous equilibrium);  $E$  and  $u$  are to be found. The question is how the emf  $\eta J$  splits into its two components  $E$  and  $u \times B$ . If, for the sake of argument, the split goes entirely one way,  $E = \eta J$ , the result is a classical (but not "classical") diffusion equation for the skin effect.

$$(13) \quad \frac{\partial B}{\partial t} = - \text{curl } E = - \text{curl } (\eta/\mu_0 \text{ curl } B)$$

with diffusion coefficients  $D_0 = \eta/\mu_0$ . In the opposite limit,  $u \times B = \eta J$ ,

$$(14) \quad \begin{aligned} u &= - \eta J \times B/B^2 \\ &= - \eta \nabla p / B^2 \end{aligned}$$

which is the "classical" diffusion rate (assuming, for simplicity, that  $u$  is perpendicular to  $B$ ). To demonstrate a diffusion equation, this velocity is inserted into mass conservation (for simplicity, isothermal)

$$(15) \quad \frac{\partial p}{\partial t} = \text{div}[(\eta p/B^2) \nabla p]$$

This is a nonlinear diffusion equation with diffusion coefficient  $D_1 = \frac{1}{2} \beta D_0$ . The actual split of  $\eta J$  between  $E$  and  $u \times B$  varies from point to point and involves solution of a boundary value problem (usually a GDE).

In a low  $\beta$ , 2-D model [8] the basic equation for diffusion of flux is

$$(16) \quad \frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = \eta \Delta \psi$$

where, from the scaling we know that  $\langle \mathbf{u} \cdot \nabla \psi \rangle = 0$  (higher order). If we were to ignore convection,

$$(17) \quad \frac{\partial \psi}{\partial t} = \eta \Delta \psi$$

we would be able to solve for  $\psi(x,y,t)$  from this equation alone. This solution would be incompatible with the equilibrium constraint,  $\Delta \psi = f(\psi)$  ( $\psi$  contours and  $\Delta \psi$  contours coincide); the term  $\mathbf{u} \cdot \nabla \psi$  is needed to maintain this constraint. The average of (16) is

$$(18) \quad \psi_t = \eta (K\psi)'$$

where  $\psi_t(V,t)$  is the derivative with  $V$  fixed and  $K(V) = \langle |\nabla V|^2 \rangle$  is a geometrical coefficient. As diffusion progresses, the change in shape of the  $\psi$ -contours is visible in (16) through  $\mathbf{u} \cdot \nabla \psi$  and in (18) through the changes in the inductance  $K(V,t)$ . The second version, in terms of  $K$ , is much more intuitive than  $\mathbf{u} \cdot \nabla \psi$ , and it is also more direct numerically. For example, for a contour of given circumference,  $K$  is a minimum for a circle. It increases with corrugations or elongation of the contour.

Figure 1 shows the history of a Belt Pinch squeezed by a rising coil current at the waist [subject to exactly

eq. (16)]. The boundary condition at the edge is chosen to eliminate the skin effect. However, the effect of corrugation of the outer surfaces is seen as a rise in current density (it would be negative, as a skin current). As the point of isolation is approached, the central contours elongate, increasing  $K$ , and causing a current peak. After isolation, a simple calculation shows that  $K$  has a logarithmic singularity at the separatrix. A simple explicit solution of (18) for a uniformly moving  $K(V-ct)$  with a logarithmic singularity closely imitates the current signature in Fig. 1. This moving current peak is seen in Doublet experiments.

Perhaps the most important single phenomenon to be verified by this numerical experiment is the fact that current penetration of a deformable medium (e.g. plasma) is not governed by the skin effect as in a solid conductor, but the current can penetrate instantaneously (anomalously!) when there is a change of shape. As a result of a cyclic external coil variation, one can even arrange to create an AC (damped sinusoidal) skin current at the center of the plasma rather than at the edge.

The significance of the difference between the  $K^2PS$  and the GH theories is sometimes confused by taking the techniques of the latter (alternating geometry and 1-D transport) and applying them to an inconsistent model which includes  $\partial B/\partial t$  and at the same time evaluates mass flow by transport formulas (e.g. [19]). It is only in the absence of  $\partial B/\partial t$  (and with



$p = p(\psi)$ ,  $T = T(\psi)$ , etc.) that mass flow can be described in terms of surface to surface transport. The overdetermination introduced by including  $\partial B/\partial t$  and a formula for mass transport is removed by including only part of Ohm's law. There is no simplification associated with this ad hoc procedure, since the correct treatment of velocity (mass flow) by the GH algorithms is at least as simple and is more accurate. The large discrepancy between K<sup>2</sup>PS "classical" mass transport and the correct classical GH mass transport in the presence of time varying fields has been calculated [12]. The same remarks, of course, apply to Neoclassical transport which does not yet include transient fields correctly (except in some special calculations such as the Ware pinch).

There is also occasional misunderstanding that the use of averaged transport is necessarily a rough approximation to the correct procedure; under appropriate circumstances (as here described), it is exact with no loss of information as demonstrated by the theory of the GDE.

One other qualitative point is the distinction between isolation and bifurcation. The necessity of island structure in plasma equilibria was pointed out in [6,20,21]. Mathematically, bifurcation (multiple solutions) can be expected when a point eigenvalue (of the dynamical operator) crosses the origin; isolation occurs (with unique continuation of the solution) when a continuum touches the origin [20,9]. Each phenomenon requires numerical care, but of an entirely different type.

The fact that a change of topology is compatible with ideal MHD [9,22] increases the importance of the adiabatic model (which, for example, gave the first demonstration of saturation of the tearing mode [9] and the first time-dependent simulation of a topology change [12]).

A detailed summary of the equations and algorithms with optional choices for various cases has been presented [14,15] and will be published. We present here a few examples of typical formulations.

#### A. Axially Symmetric Resistive Formulation (original Grad-Hogan)

$$\Delta^* \psi = -r^2 dp/d\psi - f df/d\psi, \quad f = rB_\theta$$

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = \eta \Delta^* \psi - c$$

$$\left\langle \frac{1}{r^2} \frac{\partial f}{\partial t} \right\rangle + (f \langle \mathbf{u} \cdot \nabla V / r^2 \rangle)' = (\eta K f')', \quad ' = \partial / \partial V$$

$$p_t + U p' + \gamma p U' = 0 \quad (\gamma=1 \text{ isothermal}, \gamma>1 \text{ adiabatic})$$

$$\langle \Delta^* \psi / r^2 \rangle = (K \psi')', \quad K \equiv \langle |\nabla V|^2 / r^2 \rangle$$

Note: The averaged  $\partial f / \partial t$  equation is the condition for solvability of the  $\theta$ -component of  $\mathbf{u}$  (which component appears only in the equation for  $\partial f / \partial t$ ).

A<sub>1</sub>: Formulas with V as Independent Variable

$$\langle 1/r^2 \rangle = A, \quad \langle r^2 \rangle \langle 1/r^2 \rangle - 1 = \tau^2, \quad fA = \chi', \quad U = \langle \mathbf{u} \cdot \nabla V \rangle$$

$$\psi_t + U\psi' = (\eta/A) [(K\psi')' - \tau^2 p'/\psi'] - c$$

$$\chi_t + U\chi' = \eta K(\chi'/A)' - \eta \tau^2 \chi' p'/A(\psi')^2 - c_1$$

Two interesting identities are:

$$(19) \quad (\chi_t + c_1)/\chi' - (\psi_t + c)/\psi' = \eta K f'/fA - \eta (K\psi')'/A\psi'$$

$$(20) \quad \langle B^2 \rangle_U = -\eta p' (K \langle r^2 \rangle + f^2 \tau^2 / (\psi')^2) - K\psi' (\psi_t + c) - f(\chi_t + c_1)$$

$$\langle B^2 \rangle = \langle B_p^2 + B_T^2 \rangle, \quad \langle B_p^2 \rangle = K\psi'^2, \quad \langle B_T^2 \rangle = Af^2$$

With  $\psi_t$  and  $\chi_t$  dropped (19) represents the  $K^2PS$  profile restriction and with the time derivatives dropped (20) gives the PS transport formula. To evaluate the global GH "corrections" requires complete solution of the problem. This formulation is incomplete since the quantity U has been neither determined nor eliminated.

A<sub>2</sub>: For large aspect, small poloidal field, finite poloidal  $\beta$  and rotational transform (original GH) the fast time scale evolves according to

$$\psi_t = (\eta/A) (K\psi')' - c$$

and all terms in the formula for U (including  $\psi_t$  and  $\chi_t$ ) contribute equally to the slow pressure diffusion which

accompanies the fast  $\psi$  diffusion. After  $\psi$  equilibrates, the  $\psi_t$  and  $\chi_t$  terms drop out of the expression for  $U$ , which is essentially Pfirsch-Schlüter, and gives a slow equilibration for  $p$ .

A<sub>3</sub>: For finite aspect and cross-section and finite poloidal  $\beta$  with small poloidal field, both  $p$  and  $\beta$  diffuse at the same fast rate,

$$\psi_t = (\eta/A) (K\psi')' - c$$

$$U = - \eta p' \tau^2 / A \psi'^2$$

where the modified classical  $U$  makes  $p$  diffuse as quickly as  $\psi$ . The two diffusion processes seem to be uncoupled, but they are actually coupled by the pressure balance. These formulas are misleadingly simple because the coefficients  $K$ ,  $A$ ,  $\tau^2$  vary on the same time scale as  $p$  and  $\psi$  (unless the cross-section is approximately circular).

B: Adiabatic Axially Symmetric Formulation

$$\Delta^* \psi = - r^2 \dot{p} - f \dot{f} \quad (\dot{\phantom{x}} = \partial/\partial \psi)$$

$$p = \mu(\psi) (\psi')^2, \quad f = \nu(\psi) \psi' / A \quad (\text{for } \gamma=2)$$

$$(K\psi')' = - (\dot{\mu} + \nu \dot{\nu}/A) \psi'^2 - 2\mu \psi'' - \nu^2 (\psi'/A)'$$

The adiabatically invariant profiles  $\mu(\psi)$  and  $\nu(\psi)$  are obtained by eliminating  $U$  from the equations for  $p_t$ ,  $\psi_t'$ , and  $\psi_t''$ .

The last equation (average pressure balance, with  $p$  and  $f$  removed in favor of  $\mu$  and  $v$ ) is an ordinary differential equation for  $\psi(V)$  given the adiabatic profile  $\mu(\psi)$ ,  $v(\psi)$  and the geometric inductance coefficients  $K(V)$  and  $A(V)$ . The solution procedure involves iteration on  $K(V)$  and  $A(V)$ . The 2-D elliptic operator is inverted with known right side after evaluating a profile  $\psi(V)$  from the ODE to determine a revised set of contours which redetermine  $K$  and  $A$ .

Together with a generalized definition of the concept of adiabaticity (allowing islands to form and change size, subject to appropriate jump conditions across a Separatrix), this algorithm has been used to calculate adiabatic changes of a Tokamak and adiabatic transition between a Belt Pinch and Doublet [11]. With added heat source ( $\frac{\partial \mu}{\partial t} = \text{given}$ ), it has also been used to describe a Flux Conserving Tokamak [23]. The pressure balance involves all relevant inductance coefficients, one in 2-D, two in axial symmetry, three (full  $2 \times 2$  matrix) in helical symmetry [9].

C: General Tokamak, Resistive and Heat Conducting, with Toroidal Flux,  $\chi$  as Independent Variable

To the  $\psi_t$  and  $\chi_t$  equations in  $A_1$  we add the energy equation with  $T = T(\psi)$ ,

$$p_t + Up' + \gamma p U' = (\gamma - 1) (\lambda_* T')' + (\gamma - 1) \langle \eta J^2 \rangle$$

$$\lambda_* = \lambda(T) \langle |\nabla V|^2 / \omega^2 \tau_1^2 \rangle$$

$$\langle \eta J^2 \rangle = \eta [K(f')^2 + F^2/A + \tau^2 (p')^2 / A (\psi')^2] , \quad F = (K\psi')'$$

With  $\chi$  as independent variable,

$$\phi(\chi, t): \left. \frac{\partial \phi}{\partial t} \right|_{\chi} = \frac{d\phi}{dt}, \quad \frac{\partial \phi}{\partial \chi} = \dot{\phi},$$

$$\phi_t + U\phi' = \frac{d\phi}{dt} + \dot{\phi}(\chi_t + U\chi')$$

Let  $\chi' \equiv \alpha$ ,  $\mu \equiv p/\alpha^\gamma$ ,  $v \equiv \dot{\psi}$ ,  $\xi \equiv \rho/\alpha$ .

As an abbreviation, set

$$\psi_t + U\psi' \equiv X$$

$$\chi_t + U\psi' \equiv Y$$

$$p_t + Up' + \gamma pU' \equiv Z$$

Then,

$$(C.1) \quad \begin{cases} \frac{d\mu}{dt} = \frac{1}{\alpha^\gamma} Z - \gamma \mu \dot{Y} - \dot{\mu} Y \\ \frac{dv}{dt} = (X - vY)' \\ \frac{d\xi}{dt} = -(\xi Y)' \end{cases}$$

The pressure balance takes the form

$$(C.2) \quad \alpha v(K\alpha v)' = -(\mu \alpha^\gamma)' - \alpha(\alpha/A)'$$

There are four equations for  $(\mu, v, \xi, \alpha)$ ,  $(\mu, v, \xi)$  diffuse, while  $\alpha$  satisfies an ODE. The derivatives  $\dot{\alpha}$  and  $\ddot{\alpha}$  can be explicitly eliminated in favor of  $(\dot{\mu}, \dot{v})$  and  $(\ddot{\mu}, \ddot{v})$ . In this way the second derivatives of  $(\mu, v, \xi)$  in (C.1) are explicit. The eigenvalues of the  $3 \times 3$  matrix of second derivatives determine the numerical stability and accuracy and are the

actual diffusion coefficients for small wavelength disturbances. These eigenvalues (for  $\gamma = 2$ ) are

$$\lambda_0 = \eta K \alpha^2 / A$$

$$\lambda_{\pm} = \frac{1}{2} (\lambda_1 + \lambda_2) \pm \frac{1}{2} [\lambda_1^2 + \lambda_2^2 - 2\delta \lambda_1 \lambda_2]^{1/2}$$

where

$$\lambda_1 = (2\eta \alpha^2 \mu / A \Delta) [K(1 + \tau^2) + \tau^2 / A v^2]$$

$$\lambda_2 = (\lambda_* \alpha / \xi) (1 - \mu / \Delta)$$

$$\delta = \frac{\mu}{\Delta - \mu} \quad , \quad \Delta = K v^2 + 2\mu + 1/A$$

$\lambda_0$  is the (geometry corrected) skin diffusion coefficient. There are many interesting consequences in the limits of low and high  $\beta$ , aspect, etc. Note that  $\lambda_+$  and  $\lambda_-$  are respectively larger and smaller than the larger and smaller of  $(\lambda_1, \lambda_2)$ . Also

$$\frac{\lambda_1}{\lambda_0} = \beta [1 + \tau^2 (1 + 1/A K v^2)] \quad , \quad \beta = p / \langle \frac{1}{2} B^2 \rangle$$

$\lambda_1$  is the plasma density diffusion coefficient if the heat equation is replaced by a prescribed heat source  $d\mu/dt = Q$ , or is adiabatic. In general, plasma diffusion and heat flow are coupled in  $\lambda_{\pm}$ . Nonlinearities couple all effects, even when the eigenvalues do not exhibit this.

This code has been programmed for a simple topology [at ORNL] and is being programmed for a Doublet configuration [at NYU].

D: Resistive Tokamak (no Heat Flow) with Mass as Independent Variable

The notation for  $\phi(M,t)$  ,  $M = \int \rho dx$  is

$$\left. \frac{\partial \phi}{\partial t} \right|_M = \frac{d\phi}{dt} , \quad \frac{\partial \phi}{\partial M} = \dot{\phi}$$

$$\frac{\partial \phi}{\partial t} + U\phi' = \frac{d\phi}{dt}$$

Take  $\psi$  and  $\chi$  as diffusion dependent variables and  $\rho$  as auxiliary variable:

$$\frac{d\psi}{dt} = (\eta/A) [\rho(K\rho\dot{\psi})' - \tau^2 \dot{p}/\dot{\psi}] - c$$

$$\frac{d\chi}{dt} = \eta K \rho (\rho \dot{\chi}/A)' - \eta \tau^2 \dot{\chi} \dot{p}/A \dot{\psi}^2 - c_1$$

$$\dot{\psi} (K\rho\dot{\psi})' = - \dot{p}/\rho - (\dot{\chi}/A) (\rho \dot{\chi}/A)'$$

In addition,  $p = \mu(M)\rho^\gamma$  where  $\mu(M)$  is either given or satisfies  $d\mu/dt = Q(M,t) = \text{given}$ . The third equation (ODE) can be used to eliminate  $\dot{\rho}$  and  $\dot{p}$  in favor of  $\ddot{\psi}$  and  $\ddot{\chi}$  to exhibit these second derivatives explicitly in the diffusion equations for  $\psi$  and  $\chi$ . The eigenvalues (diffusion coefficients) are  $\lambda_0$  and  $\lambda_1$  of Sec. C.

There are many satisfactory choices of variables, but they are subject to stringent restrictions. In order of differentiation (cf. pressure balance),  $p$  is comparable to  $\psi'$  and  $\chi'$ . Therefore  $\psi$  and  $\chi$  can be taken as dependent variables only if there is no energy equation (case D) or by

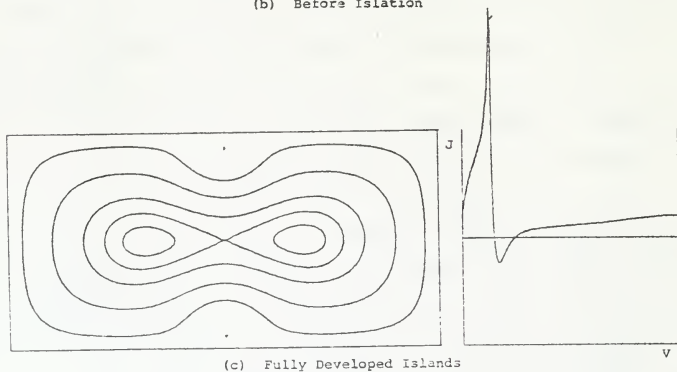
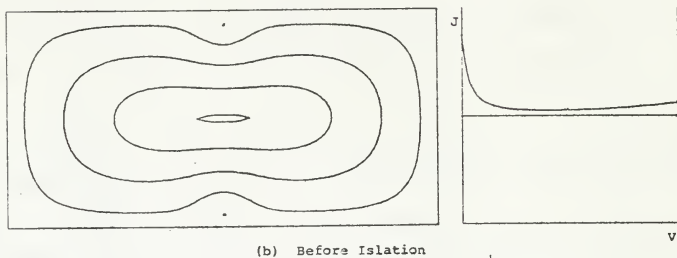
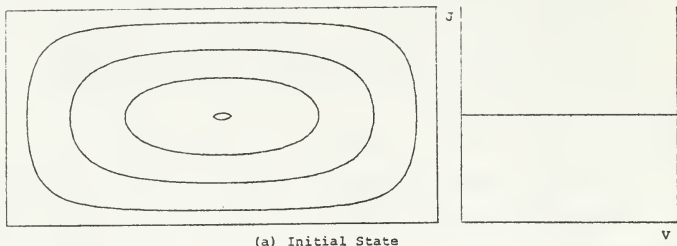


adjoining  $\int (p/\rho^\gamma) dM$  as a third variable. Or (as in case C), moving up one derivative,  $(\psi', \chi', p)$  can be taken - but only after proper normalization (to eliminate  $U$ ), viz.  $\psi'/\rho$ ,  $\chi'/\rho$ ,  $p/\rho^\gamma$ . In normalizing to eliminate  $U$ , either  $\psi'$  or  $\rho$  can be used, e.g.  $p/\rho^\gamma$  or  $p/\psi'^\gamma$ , etc. In a reversed poloidal field configuration,  $\psi$  is a poor independent variable; in a simple mirror,  $\chi \equiv 0$  cannot be used; near the edge of the plasma normalization with  $\rho$ , e.g.  $\psi'/\rho$  is unbounded, while  $p/\rho^2$  has no evident limit. In a reversed field simple mirror one has a choice between  $\psi'/\rho$  which is unbounded at the edge and  $\rho/\psi'$  which is unbounded at the separatrix. The question of whether a boundary condition is imposed, or left free, or whether a unique natural boundary condition must be imposed requires individual verification for each variable at the plasma edge, center, axis (if within the plasma), and separatrix. Many (though not all) of these permutations have been evaluated and will be reported on subsequently.

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Transition From Belt Pinch to Doublet.  
Curves of  $J$  vs.  $V$  Show Moving Singularity.

Figure 1

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